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An examination of statistical theories for fibrous materials in the light of experimental data

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We have analysed experimental data on the tensile strength of carbon fibres and bundles of parallel carbon fibres. These data are used to assess whether theoretical relations between the strengths of fibres and different kinds of bundles are consistent with experiment. The analysis confirms the presence of a hybrid effect, and also that the classical Weibull relations between strength and length are not apparently satisfied for bundles. It is suggested that the latter observation might be an effect of the random variation of fibre diameter, and some consequences of this are examined.

Notation

y	tensile stress
L, L_1, L_2	length of fibre or bundle
$F(y; L)$	failure probability (single fibre)
$f(y; L)$	density of failure stress, i.e. $\partial F/\partial y$
μ, y_1, ϱ, y_L, ϱ_L, y'_L, ϱ'_L	parameters of F
l	likelihood function
Y_i	datum (failure stress in ith specimen)
χ^2_ν	chi-square distribution with ν degrees of freedom
$\chi^2_{\nu,\alpha}$	percentage point of χ^2_ν (i.e. if X has distribution χ^2_ν then $P\{X > \chi^2_{\nu,\alpha}\} = \alpha$)
y^*, μ^*, σ^*_N	defined from $F(y, L)$, $0 \leqslant y < \infty$ (used in dry bundles analysis)
$G_N(y; L)$	failure probability for a bundle of N fibres
δ	ineffective length
m	$= L/\delta$: number of segments in chain-of-bundles model
K_k	stress concentration coefficient when k consecutive failures
q_k	number of neighbours having stress concentration K_k
k^*	critical number of consecutive failures
y_1^*, ϱ^*	Weibull parameters for bundles
α, α^*	powers of L in modified model for fibres and bundles, respectively
β	cross-sectional area of fibre
$F(y; L, \beta)$	failure probability conditional on β
μ, y_0, γ, η, c, c^*	parameters of modified model
g, g_1	densities of β and β^η
$\bar{g}$, $\bar{g}_1$	Laplace transforms of g, g_1
E	mathematical expectation

1. Introduction

There has been much interest in recent years in statistical theories for the strength of fibrous materials. The theory of loose bundles of fibres originated with Daniels [1]. A theory of unidirectional fibrous composites was started with the papers of Rosen [2] and Zweben [3]. A particular model was examined in detail by Harlow and Phoenix ([4, 5] and references therein) and in a different way by Smith [6]. Smith *et al.* [7] combined these ideas to propose a general method for calculating approximate failure

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TABLE I Summary of data

Sample	Type	Length (mm)	Number of observations	Sample mean (GPa)	Sample standard deviation (GPa)
1	Single fibres	1	57	4.24	0.85
2		10	64	3.05	0.62
3		20	70	2.45	0.49
4		50	66	2.25	0.41
5	Dry bundles	5	28	1.92	0.07
6		20	25	1.68	0.10
7		100	29	1.58	0.13
8		200	27	1.38	0.11
9	Impregnated bundles	20	28	2.82	0.16
10		50	30	2.81	0.17
11		150	32	2.68	0.19
12		300	29	2.50	0.23
13	Hybrid bundles	20	168	3.71	0.17
14		50	68	3.64	0.19
15		100	34	3.59	0.17
16		200	17	3.54	0.16

probabilities, and studied its mathematical justification in detail. Batdorf [8] proposed similar approximations. The statistical aspects of composite strength have also been examined by Bader and co-workers [9–12]. Batdorf and Ghaffarian [13] compared Batdorf's theoretical prediction with limited experimental data, but experimental and theoretical comparison has otherwise been lacking. Our purpose here is to make a more detailed comparison of some experimental data and to compare the results with theoretical predictions.

The data we examine are those of Bader and Priest [9], based on experiments which they conducted at Surrey University. A summary is in Table I. There are sixteen samples, consisting of four types of bundles each tested at four different gauge lengths. The four types are (a) single carbon fibres, (b) dry bundles of parallel carbon fibres, (c) impregnated tows of parallel carbon fibres in an epoxy resin matrix, (d) hybrid bundles consisting of tows of carbon embedded in a glass-fibre/epoxy laminate. In each case the failure load under tension was measured in an Instron testing machine, and the failure stress computed from that. For types (a)–(c), the tests were repeated independently with fibres of different lengths. For type (d), the tests were carried out on single specimens of length 200 mm, and these specimens notionally divided up to obtain data for shorter gauge lengths. Bader and Priest [9] provide much more experimental detail.

For the study of single fibres, a key concept is the "weakest-link" hypothesis that the strength of a fibre can be represented as a minimum of statistically independent strengths of sections of the fibre. It was this notion which led Weibull [14] to postulate the now well known distribution which bears his name. In spite of its wide application, the weakest-link hypothesis does not seem to have been tested very much with experimental data. In Section 2, we compare various distributions for the strength of single fibres.

In dry bundles, the fibres are arranged in parallel with no physical binding present. It therefore seems reasonable to postulate that the applied load is spread equally over the surviving fibres. The first detailed analysis of this model was due to Daniels [1]; McCartney and Smith [15] have given a recent review. In Section 3, we compare the theoretical results given in those papers with the data.

The remainder of the paper is concerned with composites, and takes the form of a comparison between the data for impregnated and hybrid composites and the theory developed in Smith *et al.* [7]. In Section 4, we summarize the main features of this theory. The results are then applied to the impregnated and hybrid bundles. For the former, there appears to be good agreement between theory and experiment. The hybrid bundles, however, are rather stronger than the theory would predict. This so-called hybrid effect has been noted by a number of earlier authors, but still does not seem to be fully explained.

The theory of Section 4 assumes that the notion of a weakest-link effect in large systems is appropriate for bundles as well as for single fibres. This assumption itself appears highly questionable. In Section 5, we consider an extension to the basic model, allowing for the random variation in fibre diameters. This goes some way towards explaining the earlier results but still leaves questions to be answered.

2. Single fibres

Let $F(y; L)$ denote the probability that a fibre of length L fails under tensile stress y. The weakest-link hypothesis may be expressed in the form

$$1 - F(y; L_1 + L_2) = [1 - F(y; L_1)] \times [1 - F(y; L_2)] \quad (1)$$

for positive y, L_1 and L_2. One solution to Equation 1 is

$$F(y; L) = 1 - \exp\{-L[(y - \mu)/y_1]^{\varrho}\} \quad (2)$$

for constants $\mu \geqslant 0$, $y_1 > 0$, $\varrho > 0$. The corresponding density is $f(y; L) = (\partial/\partial y)F(y; L)$. This was obtained by Weibull [14], though a more precise justification as one of the three limiting distributions of extreme value theory was given earlier by Fisher and Tippett [16]. This distribution is very widely used as a statistical model for the strengths of materials. In many applications μ is taken to be zero.

An alternative way of writing this theory (with $\mu = 0$) is in the form

$$F(y; L) = 1 - \exp[-(y/y_L)^{\varrho_L}] \qquad L > 0, y > 0 \quad (3)$$

together with the relations

$$\varrho_L = \varrho \qquad y_L = L^{-1/\varrho} y_1 \quad (4)$$

The first question we wish to address is whether the data actually support the weakest-link hypothesis for single fibres. Our approach to this problem will be to assume the Weibull distribution Equation 3 for each gauge length, and then to decide whether the parameters ϱ_L, y_L satisfy Equation 4. A graphical analysis on these lines was, in effect, carried out by Bader and Priest, but the difficulty of deciding whether the departures from Equation 4 are significant suggests the need for a more formal method of analysis. Our analysis will be based on the method of maximum likelihood.

The likelihood function for the model of Equations 3 and 4 may be written in the form

$$l(\varrho, y_1) = \prod_{i=1}^{n} f(Y_i; L_i) \quad (5)$$

where we have assumed a sample in n fibres, the ith being of length L_i and having failure stress Y_i. Numerical maximization of l defines the maximum likelihood estimators ϱ, y_1. The inverse of the Hessian matrix of $-\log l(\varrho, y_1)$ provides an estimate of the covariance matrix of the estimators. In particular, the square roots of the diagonal elements of that matrix are the standard errors of ϱ and y_1. Kendall and Stuart [17] is a standard reference on statistical theory including maximum likelihood.

For the data in question (Samples 1–4), this procedure leads to estimates $\varrho = 5.6$, $y_1 = 4.8$ with standard errors 0.2, 0.1 respectively. The standard error of an estimate is a measure of the uncertainty of that estimate due to sampling variation. In most practical cases the sampling distribution of an estimate is close to the normal distribution, in which case an approximate 95% confidence interval for the parameter has a width of four standard errors. In this case, for example, approximate 95% confidence intervals for ϱ and y_1 are (5.2, 6.0) and (4.6, 5.0) respectively. Bader and Priest obtained $\varrho = 5.8$ by averaging the four values from individual samples.

Now we propose a method to test the validity of Equation 4. We assume the validity of Equation 3 for each L, and test the hypothesis that the parameters y_L, ϱ_L satisfy Equation 4. Let l_0 and l_1 respectively denote the maximum values of the likelihood function Equation 5 when the parameters ϱ_L, y_L satisfy Equation 4 and when they are unconstrained. In this calculation, to obtain l_1, the likelihood function Equation 5 is maximized separately for each of the four subsamples, the product of the four maximized likelihoods being l_1. Of course $l_1 > l_0$ and the magnitude of l_1/l_0 provides a test of the Equation

TABLE II Parameter estimates and maximized log likelihoods under weakest-link hypothesis

Type	y_1 or y_1^*	Standard error	ϱ or ϱ^*	Standard error	$\log l_0$
Single fibres	4.77	0.08	5.58	0.18	−229.1
Dry bundles	2.16	0.03	14.75	0.81	70.5
Impregnated bundles	3.55	0.06	18.66	1.23	21.7
Hybrid bundles	4.31	0.02	26.4	1.09	73.3

4. The appropriate test (which is justified by asymptotic theory) is to reject Equation 4 if $2\log(l_1/l_0) > \chi^2_{6,\alpha}$, where $\chi^2_{\nu,\alpha}$ denotes the upper α-point of the distribution χ^2_ν. Here $1-\alpha$ is the level of significance which must be set by the statistician and the number $\nu = 6$ arises because this is the number of degrees of freedom imposed by the constraint Equation 4 when there are samples at four diffent gauge lengths.

For these data we find $2\log(l_1/l_0) = 18.0$, whereas $\chi^2_{6,0.01} = 16.81$. Formally, the hypothesis of Equation 4 is rejected at the 99% level of significance. Thus we are led to question whether the weakest-link hypothesis is, in fact, an appropriate assumption.

The same procedures were applied to the three types of bundles as well, with results given in Tables II and III. The χ^2 values were 50.4, 28.0 and 40.9 for the dry, impregnated and hybrid bundles, whereas $\chi^2_{6,0.001} = 22.46$. Thus, in all three cases, the hypothesis of Equation 4 is rejected at a significance level considerably greater than 99.9%. This constitutes fairly strong evidence against the weakest-link hypothesis for bundles of fibres.

The weakest-link hypothesis has been a well established feature of statistical theories of strength, so the results so far are rather hard to explain. From another point of view, Equation 4 still has good predictive power, as shown in Table IV. In this table we compare y_L from Table III, i.e. the parameter obtained just from the fibres of length L, with a number y_L^{pred} defined to be

$$y_L^{\mathrm{pred}} = y_1 L^{-1/\varrho} \tag{6}$$

where ϱ and y_1 are taken from Table II. The discrepancy between y_L and y_L^{pred} is never more than 5%, and in most cases lower. This suggests that, although there is statistical evidence against Equation 4, from a practical point of view the weakest-link relation may be justified. In later sections we shall consider these matters further.

The likelihood ratio test used here is only one of several formal tests available for choosing between models, but in moderately large

TABLE III Parameter estimates and maximized log likelihoods for individual data sets

Sample	y_L	ϱ_L	Maximum log likelihood	$\log l_1$	$2\log(l_1/l_0)$
1	4.58	5.6	−71.0		
2	3.31	5.0	−63.0		
3	2.65	5.5	−49.9	220.1	17.8
4	2.42	6.0	−36.2		
5	1.95	30.8	34.7		
6	1.72	20.9	23.1		
7	1.64	13.7	17.8	95.7	50.4
8	1.44	13.8	20.1		
9	2.90	20.8	11.8		
10	2.88	20.3	11.0		
11	2.76	19.0	10.9	35.8	28.0
12	2.61	12.9	2.0		
13	3.79	24.1	55.3		
14	3.72	22.8	19.2		
15	3.67	24.5	12.0	93.7	40.8
16	3.61	26.0	7.2		

TABLE IV Weakest-link discrepancy. Comparison of $y_L^{\text{pred}} = L^{-1/\varrho} y_1$ (ϱ, y_1 from Table II) with y_L (Table III)

Type	Length	y_L^{pred}	y_L	y_L^{pred}/y_L
Single fibres	1	4.77	4.58	1.04
	10	3.16	3.31	0.95
	20	2.79	2.65	1.05
	50	2.37	2.42	0.98
Dry bundles	5	1.94	1.95	0.99
	20	1.76	1.72	1.02
	100	1.58	1.64	0.96
	200	1.51	1.44	1.05
Impregnated bundles	20	3.02	2.90	1.04
	50	2.88	2.88	1.00
	150	2.71	2.76	0.98
	300	2.62	2.61	1.00
Hybrid bundles	20	3.84	3.79	1.01
	50	3.71	3.72	1.00
	100	3.61	3.67	0.98
	200	3.52	3.61	0.97

samples it makes little difference which one is chosen. Some comparison with graphical methods of assessment will be made in Section 4.3.

Maximum likelihood has also been used to fit two other distributions, in an effort to decide whether any other distribution fitted the data better than Equation 3. The three-parameter Weibull distribution (Equation 2) was fitted separately to all sixteen samples. A wide range of values of μ, was found, and in only two cases was the fit a significant improvement over the two-parameter fit, as assessed by the likelihood ratio test. Overall it seemed that there was no evidence in favour of this distribution.

Similar conclusions were reached for the "double-humped" Weibull distribution

$$F(y; L) = 1 - \exp\left[-(y/y_L)^{\varrho} - (y/y_L')^{\varrho'}\right] \qquad y > 0 \tag{7}$$

which has been suggested as an improvement on the standard Weibull distribution; see Harlow and Phoenix [4] for discussion. In every case, maximum likelihood estimation resulted in either $\varrho = \varrho'$ or that one of the two shape parameters is effectively $+\infty$; in either case the distribution reduces to the standard Weibull form. It should be pointed out that there are some theoretical difficulties over the application of maximum likelihood theory in this case because of the degeneracy at $\varrho = \varrho'$, but we still believe that there is no evidence in the present data to justify this distribution. Further details concerning both this and the three-parameter Weibull distributions were given by Watson [18].

3. Dry bundles

We treat the dry bundles as following Daniels' [1] model, the key feature of which is that the load is shared equally over the fibres in any configuration of failed and surviving fibres. Notation follows McCartney and Smith [15].

Daniels' basic result was that the failure stress on a bundle of N fibres is approximately normally distributed with mean and variance

$$\begin{aligned} \mu^* &= \sup_{y>0} y[1 - F(y; L)] \\ &= y^*[1 - F(y^*; L)] \quad \text{(say)} \\ \sigma_N^{*2} &= y^{*2} F(y^*; L)[1 - F(y^*; L)]/N \end{aligned} \tag{8}$$

If $F(y; L)$ is as in Equation 3, then μ^* and σ_N^* are given by

$$\begin{aligned} \mu^* &= y_L \varrho^{-1/\varrho} e^{-1/\varrho} \\ \sigma_N^* &= \mu^*[(e^{1/\varrho} - 1)/N]^{1/2} \end{aligned} \tag{9}$$

where, for notational simplicity, we have written ϱ in place of ϱ_L.

Let us illustrate this for $L = 20$ mm, the only gauge length for which both single fibre and dry bundle results are available. From Table III (Sample 3) we obtain $y_L = 2.65$, $\varrho = 5.5$ from which we calculate (for $N = 1000$) $\mu^* = 1.62$, $\sigma_N^* = 0.014\mu^*$. From Table I (Sample 6) we see that the mean bundle strength is 1.68 and we estimate the coefficient of variation as $0.062/\sqrt{25} = 0.012$, the latter value being compared with the theoretical 0.014. There seems to be remarkably good agreement.

This good agreement, however, does not extend to other values of L. By means of Equations 3 and 4, we calculate y_L and ϱ, and hence μ^* and σ^*, for the other bundle lengths tested. These are summarized in Table V. Note that the values for $L = 20$ mm differ slightly because the estimate is now based on all the single fibre data rather than just the $L = 20$ mm data. The theory does not give such good predictions for the other gauge lengths, and appears to underpredict the true strengths for $L = 100$ mm and $L = 200$ mm.

Modifications to the theory, and improved approximations, have been proposed [15, 19,

TABLE V Dry bundles

Length of fibres (mm)	Theoretical mean, μ^*	Theoretical coefficient of variance, σ_N^*/μ^*	Observed mean	Observed coefficient of variance
5	2.19	0.014	1.92	0.016
20	1.71	0.014	1.68	0.015
100	1.28	0.014	1.58	0.012
200	1.14	0.014	1.38	0.007

20], but these do not seem to account for the observed discrepancies. It is possible that the reason is the invalidity of Equation 4, but this seems unlikely to account for such large discrepancies. The most likely explanation is that there is some physical interaction between the fibres which would invalidate the theory at longer gauge lengths.

4. Impregnated and hybrid bundles

4.1. Theory

We start our discussion of composites with a short summary of the theory in Smith *et al.* [7]

Consider an array of parallel fibres whose cross-sections form a square lattice (Fig. 1). We assume that the array exhibits local stress concentrations which means, amongst other things, that the effect of fibre failures is restricted to a small length of the composite, measured in the direction of the fibre axes. This behaviour is approximated, following Rosen [2], by assuming that there is an "ineffective length" δ within which the failed fibre is unable to support load. Thus the composite is viewed as a system of statistically independent bundles, each of length δ, in series (Fig. 2).

This "chain-of-bundles" model automatically implies that the weakest-link hypothesis will hold in the following way: if $G_N(y; L)$ is the probability that a bundle of N fibres of length $L = m\delta$ fails under stress y, then we have

$$G_N(y; L) = 1 - [1 - G_N(y; \delta)]^m \quad (10)$$

We therefore concentrate on the case $L = \delta$,

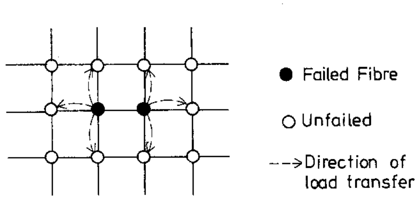

Figure 1 Square lattice configuration.

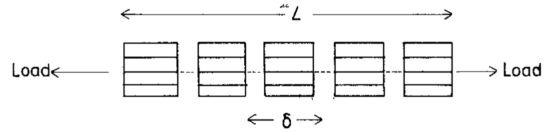

Figure 2 Chain-of-bundles model.

i.e. a single short bundle. The probability of at least one initial fibre failed under small stress y is approximately $NF(y; \delta)$, ignoring the possibility of two or more initial failures. Consider the possibility that additional failures will occur as a result of load redistribution. A failed fibre has four nearest neighbours, each of which is under a stress $K_1 y$ (say). The stress concentration factor, K_1, is between 1 and 1.25, the latter occurring if all the load from the failed fibre is absorbed by its four nearest neighbours. Therefore the conditional probability that one of the neighbours fails, given an initial failure, is approximately $4F(K_1 y; \delta)$, and so the overall probability that there are two adjacent failures in the bundle is approximately $4NF(y; \delta)$ $F(K_1 y; \delta)$. Under Equations 3 and 4 we have $F(y; \delta) \simeq \delta(y/y_1)^{\varrho}$ for small y, so this approximation becomes

$$4N\delta^2 K_1^{\varrho}(y/y_1)^{2\varrho} \quad (11)$$

This calculation may be extended to approximate the probability of k adjacent failures in the bundle as

$$\prod_{i=1}^{k-1} (q_i K_i^{\varrho}) N\delta^k (y/y_1)^{k\varrho} \quad (12)$$

where q_i is the number of maximally stressed neighbours after i consecutive failures, and K_i is the stress concentration factor on each of those neighbours.

Asymptotic results [7] suggest strongly that there is a critical value of k, henceforth denoted k^*, for which Equation 12 is a very good approximation to the bundle failure probability. A rough rule of thumb for determining this critical value is the relation

$$K_k y \simeq y_\delta \qquad (k = k^*) \quad (13)$$

since at this value the probability of failure in the neighbours approaches 1.

Combining Equations 10 and 12 and using the limit $(1 - x/m)^m \to e^{-x}$ as $m \to \infty$ leads to

$$G_N(y; L) \simeq 1 - \exp[-L(y/y_1^*)^{\varrho^*}] \quad (14)$$

where

$$\varrho^* = k^* \varrho$$

$$y_1^* = N^{-1/\varrho^*} \left(\prod_{i=1}^{k^*-1} q_i^{1/\varrho} K_i \right)^{-1/k^*} \delta^{1/\varrho^* - 1/\varrho} y_1 \tag{15}$$

In particular, these relations imply that composite strengths should also follow a Weibull distribution but with increased shape parameter (and hence decreased coefficient of variation).

Smith *et al.* [7] assumed a hexagonal array and developed detailed approximations which essentially agree with those described here. In this account we have not distinguished between impregnated and hybrid bundles but it is known, for example, that the matrix properties affect the ineffective length and the stress concentration factors, and this is one way in which the difference might be manifested.

4.2. Data analysis

We first fit the model defined by Equations 3 and 4 to single fibres and estimate the parameters ϱ and y_1, assuming the correctness of the model. Then, taking impregnated and hybrid bundles separately, we estimate the corresponding parameters, now denoted ϱ^* and y_1^*, for bundles (Table II). Our objective is to decide whether, allowing for experimental and sampling error, the results are consistent with Equation 15.

For impregnated bundles, we estimate y_1^*/y_1 to be 0.74 from Table II. The standard error may be calculated as approximately 0.02. Similarly, ϱ^*/ϱ is estimated to be 3.34 with standard error approximately 0.24.

We assume the values of q_i and K_i derived in Smith *et al.* [7] from an artificial mechanical load-sharing rule (MLLS). A method of calculating exact stress concentrations was given elsewhere [21], but explicit numerical results are available only for a few special cases involving small numbers of breaks The artificial MLLS is believed to overestimate the stress concentration factors, but not sufficiently to have a great effect on the anlysis. The other unknown constant in the theory is δ. We have taken $\delta = 0.04$ mm following a rule of thumb that δ is usually about five times the mean fibre diameter (here 8 μm), but this is really just a guess and major source of uncertainty.

For the impregnated bundles, we guess that k^* is 3 or 4 and estimate y_1^*/y_1 from Equation 15 to be 0.73 if $k^* = 3$, or 0.67 if $k^* = 4$. This is in very good agreement with the data-determined value 0.74.

For hybrid bundles we do not get such good agreement. From Table II we estimate $y_1^*/y_1 = 0.90$, $\varrho^*/\varrho = 4.66$, with standard errors 0.02 and 0.06, respectiely. We guess that k^* is 4 or 5 and estimate y_1^*/y_1 from Equation 15 as 0.75 ($k^* = 4$) or 0.68 ($k^* = 5$). These are well under the data-determined 0.90. Moreover, this assessment is not changed by making small adjustments to be assumed values of K_i and δ. For example, if we changed δ (keeping everything else the same) we would have to take $\delta = 0.01$ mm to obtain agreement between theory and experiment. Since this is only just over the mean fibre diameter, it does not appear realistic. Therefore, we are forced to conclude that there is a real "hybrid effect" leading y_1^* to be about 20 to 25% larger than would otherwise be the case.

In this analysis we have assumed the validity of Equation 4. Our conclusions therefore depend very much on how we assess the discrepancies observed in Section 2. It appears from Table IV that the discrepancy resulting from Equation 4 is less than 5%; this is comparable with the discrepancy observed in Equation 15 for impregnated bundles, and much less than that for hybrid bundles. Thus, it appears that the discrepancies from Equation 4 are not significant when comparing single fibres with bundles, though they are significant when comparing fibres of different lengths. In view of this, we believe that our method of analysis is justified. Nevertheless, there remains the possibility that an alternative model may explain all the data better, and we shall propose one in Section 5.

The hybrid effect has already been noted by Manders and Bader [10, 11]. They refer to earlier work suggesting the possible explanation that there is a change in the basic fracture mechanics: y_1 for a fibre embedded in laminate may be larger than for a loose fibre. This could explain our results, but our results do not provide any confirmation that this is the correct explanation. There is also theoretical work on probabilistic models for hybrids; Fukuda [22] gives many references to earlier work, and an alternative recent approach is that of Harlow [23]. Fukuda and Harlow, however, considered a model in which the two constituent fibres are completely

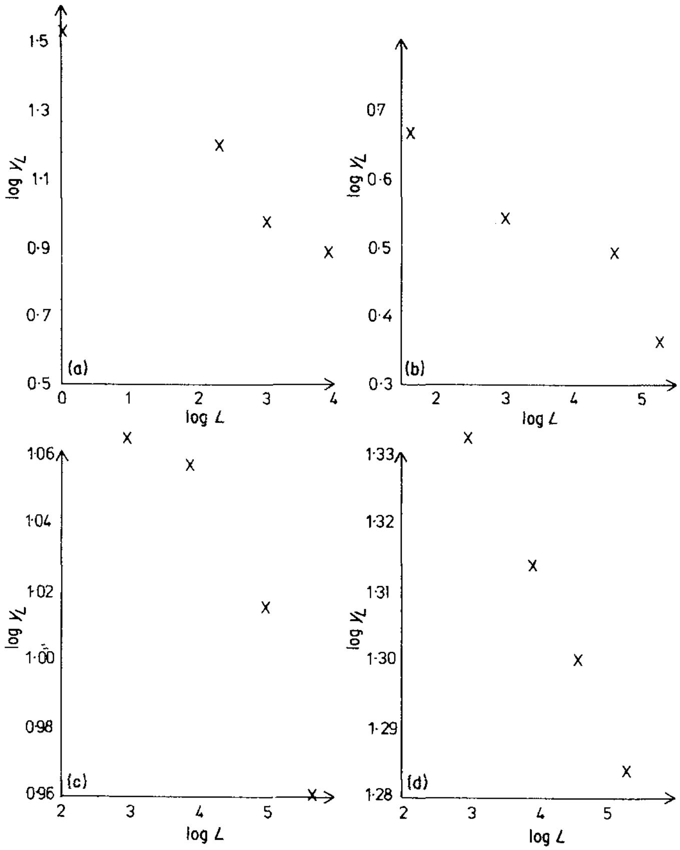

Figure 3 Plots of log y_L against log L for (a) single fibres, (b) dry bundles, (c) impregnated bundles and (d) hybrid bundles.

mixed, whereas the hybrids being studied here consist of separate phases of carbon and glass.

4.3. Comparison with graphical analysis

Bader and Priest [9] analysed the data graphically using two plotting techniques. The first was a standard Weibull probability plot, done separately for each sample. The second was a plot of log strength against log gauge length, which under the Weibull weakest-link model should be a straight line (see Fig. 3). In either case, the Weibull shape parameter may be estimated as the negative reciprocal of the slope of the plot. The first technique, probability plotting, produced results very similar to those in Table II. The second technique led to estimates of $\varrho = 6.1$ for single fibres, $\varrho^* = 27$ for impregnated bundles and 45 for hybrid bundles. Thus there is good agreement with the other estimates for single fibres, but not for bundles. Bader and Priest concluded that "neither type of bundle meets the strength/length relationship implicit in the Weibull model", a conclusion consistent with the results of our likelihood ratio tests in Section 2. These results cast further doubt on the weakest-link relation, and reinforce the need to consider alternative models.

5. Re-examination of the weakest-link hypothesis

We have observed, on the basis of our statistical analysis, that the weakest-link hypothesis implicit in Equations 3 and 4 may not be valid for bundles, even though it appears plausible for single fibres. Watson [18] used both graphical and likelihood-based methods to compare a number of parametric statistical models. One plot is reproduced as Fig. 3, in which log y_L is plotted against log L for each of the four types of material. The relationship is approximately linear, but with slope significantly different from $-1/\varrho$ or $-1/\varrho^*$ taken from Table II. This suggested the model

$$G_N(y;\, L) \;=\; 1 - \exp[-L^{\alpha^*}(y/y_1^*)^{\varrho^*}] \tag{16}$$

Of course, if $\alpha^* = 1$ this reduces to the model considered previously. The results of fitting

TABLE VI Parameter estimates and maximized log likelihoods under model $G(y; L) = 1 - \exp[-L^{\alpha}(y/y_0)^{\varrho}]$

Type	α	y_0	ϱ	Maximum log likelihood ($\log l_2$)	$T_1 = 2\log(l_1/l_2)$	$T_2 = 2\log(l_2/l_0)$
Single fibres	0.90	4.63	5.31	−227.6	15.0	2.75
Impregnated bundles	0.58	3.25	16.83	29.6	12.4	15.6
Hybrid bundles	0.48	4.02	23.92	93.4	0.7	40.1

Equation 16 are given in Table VI. Single fibres (writing α instead of α^*) are included in this in order to provide a better comparison with the results for bundles. Also given in Table VI are the test statistics $T_1 = 2\log(l_1/l_2)$ and $T_2 = 2\log(l_2/l_0)$. These are the appropriate likelihood ratio statistics for (a) testing the model of Equation 16 against the alternative that ϱ_L and y_L are arbitrary, and (b) testing the model of Equation 4 against Equation 16. The corresponding degrees of freedom are 5 and 1 respectively. Comparing these statistics with the percentage points of the chi-square distribution ($\chi^2_{5,0.01} = 15.1$, $\chi^2_{1,0.01} = 6.63$), we conclude that the improvement as we pass from Equations 4 to 16 is not statistically significant in the case of single fibres (i.e. the improvement could be due to chance variation), but very definitely significant for both impregnated and hybrid bundles.

If Equation 16 holds, then for each L the Weibull shape parameter is ϱ^*, as before. On the other hand, a log strength against log length plot will have slope $-\alpha^*/\varrho^*$. Since α^* is about 0.5, the shape parameter estimated from this plot will be about twice that estimated from the probability plots. This is consistent with the conclusions of Bader's and Priest's graphical analysis.

We now attempt an explanation of how Equation 16 might come about. Consider single fibres first. It is known that there is appreciable variation in the diameters, and hence cross-sectional areas, of fibres. For a fibre with cross-sectional area β, Equation 2 may be replaced by

$$F(y; L, \beta) = 1 - \exp\{-L\beta[(y - \mu)/y_0]^{\gamma}\} \tag{17}$$

for constants μ, y_0 and γ. From now on, we take $\mu = 0$. If β is random with probability density function (pdf) $g(\beta)$, then we have

$$\begin{aligned} F(y; L) &= 1 - \int_0^{\infty} \exp[-L\beta(y/y_0)^{\gamma}]g(\beta)\,\mathrm{d}\beta \\ &= 1 - \bar{g}[(L(y/y_0)^{\gamma}] \end{aligned} \tag{18}$$

where $\bar{g}$ denotes the Laplace transform of g. Equations 16 and 18 are consistent provided

$$\bar{g}(t) = \exp(-ct^{\alpha}) \qquad t > 0 \tag{19}$$

for positive c and α. In this case we have $\varrho = \alpha\gamma$, $y_1 = y_0 c^{-1/\varrho}$.

It is possible for Equation 19 to hold exactly, provided $\alpha < 1$. This will be the case if β has a positive stable distribution with index α ([24], Section XIII.6). This does not seem very plausible, however. It would imply that β has infinite mean and hence, in particular, may take on arbitrarily large values. Also, such a theory is not easily extended to bundles. More satisfactory results are obtained if we assume only that Equation 16 holds for large L (compared with δ) and hence that Equation 19 holds in some asymptotic sense as $t \to \infty$. The sense we shall consider is that

$$t^{-\alpha}\log\bar{g}(t) \to -c \qquad \text{as } t \to \infty \tag{20}$$

De Bruijn's [25] Tauberian theorem (for background information see Section 4.15 of [26]) shows that Equation 20 is equivalent to

$$\beta^{\alpha}[-\log g(\beta)]^{1-\theta} \to c\alpha^{\theta}(1 - \alpha)^{1-\alpha} \qquad \text{as } \beta \to 0 \tag{21}$$

Note that Equation 21 leaves $g(\beta)$ unrestricted for large β. Thus we have the conclusion: if $g(\beta)$ satisfies Equation 21 then Equation 16 will hold (writing α in place of α^*) asymptotically for large L.

The above accounts for single fibres. Now let us turn to bundles. We assume the chain-of-bundles model as before, and consider a single bundle of length δ. If fibre j has cross-section β_j then the probability of at least one failure in the bundle, under stress y, is approximately $\Sigma_j \delta\beta_j(y/y_0)^{\gamma}$. The calculations of conditional probabilities of adjacent fibre failures will proceed much as before with one important modification: the stress concentration factors are now random, since it is load rather than stress which is redistributed and the loads on individual fibres depend on their diameters. As an *ad hoc* method of taking this into account, we modify Equation 12 to

$$\left(\prod_{i=1}^{k-1} (q_i K_i^\gamma)\right)\left(\sum_{j=1}^{N} \beta_j^\eta\right) \delta^k (y/y_0)^{k\gamma} \quad (22)$$

for some η. This reflects the notion that the local stress concentrations near fibre j will depend on β_j in some way, which we are taking to follow a power law.

Assuming Equation 22 in place of Equation 12, we obtain

$$1 - G_N(y; L)$$
$$\simeq E\left\{\exp\left[-(L/\delta)\delta^{k^*}\left(\prod_{i=1}^{k^*-1} (q_i K_i^\gamma)\right)\right.\right.$$
$$\left.\left.\times (y/y_0)^{k^*\gamma} \sum_j \beta_j^\eta\right]\right\} \quad (23)$$

where E stands for expectation with respect to the distribution of $\beta_1, \ldots, \beta_N$. Now, if Equation 21 holds for the density of β, then a similar relation holds for the density of β^η. Reversing the steps which led from Equation 20 to 21, if $\bar{g}_1$ is the Laplace transform of the density of β^η, i.e.

$$\bar{g}_1(t) = \int_0^\infty \exp(-t\beta^\eta) g(\beta)\,\mathrm{d}\beta \quad (24)$$

then

$$t^{-\alpha^*} \log \bar{g}_1(t) \to -c^* \qquad \text{as } t \to \infty \quad (25)$$

where $\alpha^* = \alpha/[\alpha + \eta(1 - \alpha)]$ and c^* is related to c by

$$(1 - \alpha^*)(\alpha^*)^{\alpha^*/(1-\alpha^*)}(c^*)^{1/(1-\alpha^*)}$$
$$= (1 - \alpha)\alpha^{\alpha/(1-\alpha)} c^{1/(1-\alpha)} \quad (26)$$

Combining Equations 23 and 25 leads to the asymptotic relation

$$-\log[1 - G_N(y; L)]$$
$$\sim Nc^* L^{\alpha^*} \delta^{\alpha^*(k^*-1)}\left(\prod_{i=1}^{k^*-1} q_i K_i^\gamma\right)^{\alpha^*}$$
$$\times (y/y_0)^{\alpha^* k^* \gamma} \quad (27)$$

which is consistent with Equation 16 for

$$\varrho^* = \alpha^* k^* \gamma$$

$$y_1^* = N^{-1/\varrho^*}\left(\prod_{i=1}^{k^*-1} q_i K_i^\gamma\right)^{-\alpha^*/\varrho^*}$$
$$\times \delta^{\alpha^*/\varrho^* - 1/\gamma} (c^*)^{-1/\varrho^*} y_0 \quad (28)$$

We now examine the consequences of this for the data. The parameters y_0 and c are related by $y_1 = y_0 c^{-1/\varrho}$ but are not separately identifiable, so there is no loss of generality in taking $c = 1$. In the single fibre analysis we have $\alpha = 0.90$, $y_0 = 4.63$, $\varrho = 5.3$, $\gamma = 5.9$. For the impregnated bundles we have $\alpha^* = 0.58$ and consequently estimate η to be about 6. Also $\varrho^*/(\alpha^*\gamma) = 4.9$, suggesting k^* is 5. Equation 28 then leads us to predict $y_1^*/y_1 = 0.65$, compared with the data-determined value of 0.70. For the hybrid bundles the corresponding figures are $\alpha^* = 0.48$, $\eta = 9$, $\varrho^*/(\alpha^*\gamma) = 8.4$ (suggesting $k^* = 8$), predicted $y_1^*/y_1 = 0.69$ compared with 0.87. Again there is reasonable agreement for the impregnated bundles but a definite hybrid effect for the hybrid bundles.

To summarize, the theory developed in this section provides a possible explanation of the failure of the weakest-link hypothesis. The new theory predicts bundle strengths about as well as the old theory, with a definite but unexplained hybrid effect.

It is only a conjecture that the variation of fibre diameter is important since we have not examined any data on this feature, but there is some independent evidence. Phoenix *et al.* [27] studied commercial Kevlar fibres in which they related strengths to the linear densities of the fibres. Their results are not directly relevant in view of the different material involved, but they did note significant variations in density from fibre to fibre, and argued that it was important to take this into account.

6. Summary and conclusions

In this paper we have employed the method of maximum likelihood for the estimation of Weibull parameters. The advantages of maximum likelihood over probability plotting techniques may be summarized as (i) complex statistical models may be fitted, (ii) standard errors are computable, and (iii) formal testing procedures are available for discriminating between models. On the other hand, graphical methods are useful for assessing model fit and for detecting anomalous observations, and a thorough statistical analysis should combine both graphical and more formal analytic techniques.

In Sections 3 and 4, we compared the theoretical and experimental results. The agreement was good for impregnated bundles and we observed clear evidence of a positive "hybrid effect" in the hybrid bundles. An important issue in these comparisons is the validity of the weakest-link hypothesis. Both graphical and analytical tests cast doubt on this hypothesis, though from a predictive point of view the discrepancy did not seem too great. Further analysis in Section 5 led to an improved model, and we concluded by outlining a possible, but tentative, explanation for this new model.

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